A Picard theorem for the Askey-Wilson operator[1]

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Outline

$q-$shifted factorials

- The $q-$shifted factorials are defined by

$$(a;\, q)_0 := 1; \quad (a;\, q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}), \quad n = 1, 2, \cdots$$

- multiple $q-$shifted factorials is defined by

$$(a_1,\, a_2,\, \cdots,\, a_k;\, q)_n := \prod_{j=1}^{k}(a_j;\, q)_n. \tag{1}$$

- Without loss of generality, we may assume that $|q| < 1$ henceforth. Thus, the infinite product

$$(a_1,\, a_2,\, \cdots,\, a_k;\, q)_\infty = \lim_{n\to+\infty}(a_1,\, a_2,\, \cdots,\, a_k;\, q)_n$$

always converge.

L. J. Rogers' generating functions I

- In a well-known paper of Askey & Ismail in 1983, they gave the weight function of *continuous* $q-$*Hermite polynomials* generated by Rogers in 1894:

$$f(x)=\frac{1}{(te^{i\theta},\,te^{-i\theta};\,q)_\infty}=\sum_{k=0}^{\infty}\frac{H_k(x\,|\,q)}{(q;\,q)_k}\,t^k,\quad |t|<1,$$

where

$$H_n(x\,|\,q)=\sum_{k=0}^{n}\frac{(q;\,q)_n}{(q;\,q)_k\,(q;\,q)_{n-k}}\,e^{i(n-2k)\theta},\quad x=\cos\theta.$$

- The *poles* of $f(x)$ are enumerated by the infinite sequence

$$x_n:=\frac{1}{2}\big(q^{1/2+n}+q^{-1/2-n}\big)\ \longmapsto\ 0,\qquad n\in\mathbb{N}\cup\{0\}.$$

L. J. Rogers' generating functions II

- The same paper also gave a weight of *continuous* $q-$*ultraspherical polynomials* generated by Rogers:

$$H(x) := \frac{(\beta e^{i\theta}t,\ \beta e^{-i\theta}t;\ q)_\infty}{(e^{i\theta}t,\ e^{-i\theta}t;\ q)_\infty} = \sum_{n=0}^{\infty} C_n(x;\ \beta\,|\,q)\,t^n, \quad x = \cos\theta,$$

where

$$C_n(x;\ \beta\,|\,q) = \sum_{k=0}^{n} \frac{(\beta;\ q)_k(\beta;\ q)_{n-k}}{(q;\ q)_k(q;\ q)_{n-k}} \cos(n-2k)\theta$$

The *pole-sequence* is as on last page while the *zero-sequence* of the $H(x)$ is given by:

$$x_n := \frac{1}{2}\big(\beta t\, q^n + q^{-n}/(\beta t)\big) \longmapsto 0, \qquad n \in \mathbb{N}\cup\{0\}. \quad (2)$$

Conventional view

- They are related to the proof of Rogers-Ramanujan identities by Rogers
- It is obvious that the above generating functions have *infinitely many zeros/poles* in $\mathbb{C}$ of the forms:

$$x_n := \frac{1}{2}\left(z_a\, q^n + q^{-n}/z_a\right), \quad n \in \mathbb{N} \cup \{0\}.$$

- We shall argue that the two generating functions, etc. are *zero/pole scarce* when *interpreted appropriately*.
- Need a *difference operator* for which these "zeros/poles" belong.
- Then we built a complex function theory around this operator for which the zero/poles sequences considered can be interpreted suitably.
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Little Picard's Theorem

- Theorem (Picard showed in 1879)

  An entire function f assumes every value in $\mathbb{C}$, except perhaps for at most one exception

  (E.g. $f(x) = e^x$.)

  - Method: Elliptic modular functions and Liouville's theorem.
  - Thus for an non-constant meromorphic function f

  $$f(\mathbb{C}) = \hat{\mathbb{C}} \setminus \{\text{at most two points}\}.$$

  That is, a meromorphic function that omits three points must reduce to a *constant*.

  - We say points in $\hat{\mathbb{C}}$ that are missed or assumed only finitely many times by f a Picard exceptional values.

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Nevanlinna Characteristic fn

- **Nevanlinna characteristics**

$$\begin{aligned} T(r,f) &:= m(r,f) + N(r,f) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta + \int_0^r \frac{n(t,f)}{t} dt. \\ &= \text{(Proximity fn)} + \text{(Integrated counting fn)} \end{aligned}$$

- $$n(r,f) := \#\,\{\text{poles of } f(z) \text{ in } |z| < r\}, \quad \log^+ \xi := \max\{0, \log \xi\}.$$

- Abbreviation: for arbitrary $a \in \mathbb{C}$

$$N(r,a) = N(r, \frac{1}{f-a})$$

- $T(r,f)$ is a convex function of $\log r$, $T(r,f) \uparrow \infty$ as $r \uparrow \infty$.

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$$N(r,\,a) = N\Big(r,\,\frac{1}{f-a}\Big)$$

- $T(r,\,f)$ is a convex function of $\log r$, $T(r,\,f)\uparrow\infty$ as $r\uparrow\infty$.

Examples

- $$T(r, e^z) \sim r, \quad \sigma(e^z) = 1$$

- Let $\Gamma(z)$ denote the standard Euler-Gamma function

$$1/\Gamma(z) = ze^{\gamma} \prod_{n=1}^{+\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where $\gamma = 0.5772\ldots$. Then we have

$$T(r, \Gamma) \sim r \log r, \qquad \sigma(\Gamma) = 1,$$

- Let f be a meromorphic function, then f is transcendental if and only if

$$\liminf_{r \to +\infty} \frac{T(r, f)}{\log r} = +\infty.$$

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Nevanlinna Theory I

- Key inequality I: Given $a_1,\, a_2 \in \mathbb{C}$,

$$\begin{aligned} T(r,\,f) < N(r,\,f) + N(r,\,a_1) + N(r,\,a_2) - N_1(r,\,f) \qquad (3)\\ + O\big(r\log T(r,\,f)\big), \quad r\to\infty\ (\notin E) \end{aligned}$$

where

$$N_1(r,\,f) = N(r,\,1/f') + 2N(r,\,f) - N(r,\,f').$$

- z_0 is a pole of f:

$$\begin{aligned} \text{contrib. of } N(r,\,f) - N_1(r) &= N(r,\,f) - 2N(r,\,f) + N(r,\,f') \\ &= -N(r,\,f) + N(r,\,f') = 1; \end{aligned}$$

- z_0 is a a_j-point $(j = 1,\,2)$ of f:

$$\begin{aligned} \text{contrib. of } N(r,\,a_j) - N_1(r) &= N(r,\,a_j) - N(r,\,1/f') \\ &= N(r,\,a_j) - N(r,\,1/(f-a_j)') = 1; \end{aligned}$$

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- Key inequality II: Given $a_1,\ a_2 \in \mathbb{C}$,

$$\begin{aligned} T(r,\,f) < \overline{N}(r,\,f) + \overline{N}(r,\,a_1) + \overline{N}(r,\,a_2) \\ + O\big(r\log T(r,\,f)\big), \quad r\to\infty\ (\notin E) \end{aligned} \tag{4}$$

where

$$\begin{aligned} \overline{N}(r,\,f) &= \text{counts each pole with multiplicity } 1, \\ \overline{N}(r,\,a_j) &= \text{counts each } a_j\text{-point with multiplicity } 1 \end{aligned}$$

- Multiply $\frac{-1}{T(r,f)}$ and add 3 on both sides:

$$\Big(1-\frac{\overline{N}(r,\,f)}{T(r,\,f)}\Big)+\Big(1-\frac{\overline{N}(r,\,a_1)}{T(r,\,f)}\Big)+\Big(1-\frac{\overline{N}(r,\,a_2)}{T(r,\,f)}\Big)+o(1) \le 3-1$$

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- If f misses ∞, a_1, a_2, then the above becomes

  $$3+o(1)\approx(1-o(1))+(1-o(1))+(1-o(1))\le 2.$$

  A contradiction and thus proves the Little Picard Theorem.

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Difference Variations

- There are many generalisations to higher dimensional spaces $\mathbb{C}^n$ where Picard values are replaced by appropriate varieties.
- We re-interpret the followings:

(i) **constants** belong to $\ker\left(\frac{d}{dx}\right)$

(ii) f has three Picard values a, b, c means

$$f^{-1}(a) = \emptyset, \quad f^{-1}(b) = \emptyset, \quad f^{-1}(c) = \emptyset.$$

$\rightarrow$

(I) **functions** belong to $\ker$ of a difference operator

(II)

$$f^{-1}(a) \neq \emptyset, \quad f^{-1}(b) \neq \emptyset, \quad f^{-1}(c) \neq \emptyset.$$

but each lies on a *specific sequence*.

- Halburd-Korhonen (2006), Chiang-Feng (2008, 2018), Cheng-Chiang (2017), Chiang-Luo (2017).

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Askey-Wilson difference operator

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$$x = \cos\theta = \frac{1}{2}(z + \frac{1}{z}) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \quad z = e^{i\theta}.$$

- The *AW-divided difference operator* (1985) is defined by

$$\big(\mathcal{D}_q f\big)(x) := \frac{f(\hat{x}) - f(\check{x})}{\hat{x} - \check{x}}, \quad |q| \neq 1 \tag{5}$$

where

$$\hat{x} := \frac{q^{1/2}z + q^{-1/2}z^{-1}}{2}, \quad \check{x} := \frac{q^{-1/2}z + q^{1/2}z^{-1}}{2}.$$

- In fact the denominator above can be rewritten as

$$(q^{1/2} - q^{-1/2})i\sin\theta.$$

- If f is differentiable at x then $\big(\mathcal{D}_q f\big)(x) \to f'(x)$ as $q \to 1$.

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Logarithmic Derivative estimates

- Let $P(x)$ be a polynomial. Then

$$\int_0^{2\pi} \log^+ \left| \frac{P'(re^{i\theta})}{P(re^{i\theta})} \right| d\theta \to 0, \quad r \to \infty.$$

- The crucial tool behind the Fundamental inequalities is that the above estimate continue to hold in the following sense:

$$\begin{aligned} m\left(r, \frac{f'(z)}{f(z)}\right) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \\ &= O\big(\log T(r, f)\big) \\ &= o\big(T(r, f)\big) \end{aligned}$$

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$q-$Logarithmic Difference Lemma

- Recall that f has *finite order* σ if $\forall\varepsilon > 0$,

$$T(r, f) = O(r^{\sigma+\varepsilon}), \quad r \to +\infty.$$

If f has zero-order, then we say f has *finite log-order* $\sigma_{\log}$ when $\forall\varepsilon > 0$,

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- Theorem (C. and Feng (2018) logarithmic difference lemma)
Let $f(x)$ be a meromorphic function s.t. $\mathcal{D}_q \not\equiv 0$ and of log-order $\sigma_{\log} < \infty$ and where $|q| \neq 1$. Then we have $\forall\varepsilon > 0$,

$$m\Big(r, \frac{(\mathcal{D}_q f)(x)}{f(x)}\Big) = O\Big((\log r)^{\sigma_{\log}-1+\varepsilon}\Big) \tag{6}$$

holds for all $|x| = r > 0$.

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AW-Nevanlinna Theory I

- Key inequality I': Given $a_1,\ a_2 \in \mathbb{C}$. The log-difference lemma above leads to

$$T(r,\,f) < N(r,\,f) + N(r,\,a_1) + N(r,\,a_2) - N_{\mathrm{AW}}(r,\,f) \qquad (7)$$
$$+\, O\Big((\log r)^{\sigma_{\log}-1+\varepsilon}\Big), \qquad r \to \infty$$

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$$N_{\mathrm{AW}}(r,\,f) = N(r,\,1/\mathcal{D}_q f) + 2N(r,\,f) - N(r,\,\mathcal{D}_q f).$$

- The main task here is to find an analogue $\widetilde{N}_{\mathrm{AW}}(r,\,f)$ for $\overline{N}(r,\,f)$ for the AW-operator $\mathcal{D}_q$.

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- Our aim is to find a correct $\widetilde{N}_{\mathrm{AW}}(r, f)$ so that

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  where the AW-*integrated counting fns* are defined by

- $$\widetilde{N}_{\mathrm{AW}}(r, a) = \widetilde{N}_{\mathrm{AW}}\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{\tilde{n}_{\mathrm{AW}}(t, a)}{t}\, dt,$$

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AW-type $a-$points counting functions I

We define the *Askey-Wilson-type counting function* of f

$$\tilde{n}_{\mathrm{AW}}(r, a) = \tilde{n}_{\mathrm{AW}}\left(r, \frac{1}{f-a}\right) = \sum_{\substack{|x|<r, \\ h= \text{ multiplicity of } f(x)=a, \\ k= \text{ multiplicity of } \mathcal{D}_q f(\hat{x})=0}} (h-k)$$

over all x in $\{|x| < r\}$ where $h = h(x)$ is the *multiplicity* of the $a-$points of $f(x)$, and $k = k(x)$ is the *multiplicity* of the $0-$point of $\mathcal{D}_q f(\hat{x})$, respectively.

AW-type pole counting functions II

Similarly, we define

$$\tilde{n}_{\mathrm{AW}}(r,\infty) = \tilde{n}_{\mathrm{AW}}\left(r, \frac{1}{f}=0\right) = \sum_{\substack{|x|<r,\\ h=\text{ multiplicity of } 1/f(x)=0,\\ k=\text{ multiplicity of } \mathcal{D}_q(1/f)(\hat{x})=0}} (h-k)$$

over all x in $\{|x|<r\}$, where $h=h(x)$ is the *multiplicity* of the zeros of $1/f(x)$, and $k=k(x)$ is the *multiplicity* of zeros of $\mathcal{D}_q(1/f)$ at the $\hat{x}$.

AW-Nevanlinna Deficiency

- We have

$$0 \leq \Theta_{\mathrm{AW}}(a) = 1 - \overline{\lim_{r\to\infty}} \frac{\tilde{N}_{\mathrm{AW}}(r,\, A)}{T(r,\, f)} \leq 1$$

We call a complex number $a \in \mathbb{C}$ an

- $\mathrm{AW}-$*Picard value* if $\tilde{n}_{\mathrm{AW}}(r,\, a) = O(1)$ (equivalent to $\widetilde{N}_{\mathrm{AW}}(r,\, a) = O(\log r)$),
- $\mathrm{AW}-$*Nevanlinna deficient value* if $\Theta_{\mathrm{AW}}(a) > 0$.
- If a is a $\mathrm{AW}-$*Picard value*, then $\Theta_{\mathrm{AW}}(A) = 1$, and

$$x_n := \frac{1}{2}(z_a q^n + q^{-n}/z_a), \quad n \in \mathbb{N} \cup \{0\}.$$

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AW-Picard theorem

Theorem (C. & Feng (2018))

Let f be a meromorphic function with logarithmic order $\sigma_{\log} < \infty$, and that f has three distinct $\mathrm{AW}-$Picard exceptional values. Then f is an $\mathrm{AW}-$constant.

Proof.

We deduce (skipping details) that

$$3 = \Theta_{\mathrm{AW}}(\infty) + \Theta_{\mathrm{AW}}(a_1) + \Theta_{\mathrm{AW}}(a_2) \leq 2.$$

$\square$

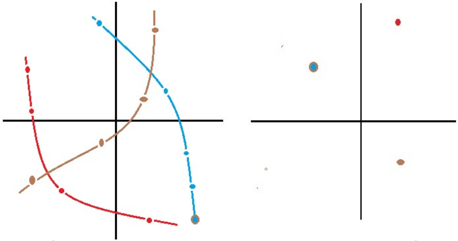

Figure: The left-side contains the pre-images of the right-side

Meromorphic fn with Extremal Deficiency

- Recall the weight of *continuous* $q-$*ultraspherical polynomials* discovered by Rogers:

$$H(x) := \frac{(\beta e^{i\theta}t,\, \beta e^{-i\theta}t;\, q)_\infty}{(e^{i\theta}t,\, e^{-i\theta}t;\, q)_\infty} = \sum_{n=0}^{\infty} C_n(x;\, \beta\,|\, q)\, t^n, \quad x = \cos\theta,$$

- The zero and pole sequences are

$$x_n = \frac{1}{2}\big(\beta t\, q^n + q^{-n}/(\beta t)\big), \quad x_n := \frac{1}{2}\big(q^{1/2+n} + q^{-1/2-n}\big)$$

$n \in \mathbb{N} \cup \{0\}$ respectively.

- $$\Theta_{\mathrm{AW}}(0) = 1, \quad \Theta_{\mathrm{AW}}(\infty) = 1.$$

Thus $\Theta_{\mathrm{AW}}(0) + \Theta_{\mathrm{AW}}(\infty) = 2$ which is the maximal deficiency sum without the $H(z)$ being in the kernel of $\mathcal{D}_q$.

General Main theorem

- Theorem (C. & Feng (2018))

Suppose that $f(z)$ *is a non-constant meromorphic function of log-order* $\sigma_{\log} < \infty$. *Let* q *be a complex number such that* $|q| \neq 1$, $\mathcal{D}_q f \not\equiv 0$, *and let* $a_1, a_2, \cdots, a_p$ *where* $p \geq 2$, *be mutually distinct elements in* $\mathbb{C}$, *then we have for* $r < \infty$ *and for every* $\varepsilon > 0$

$$(p-1+o(1))\, T(r, f) \leq \widetilde{N}_{\mathrm{AW}}(r, f) + \sum_{\nu=1}^{p} \widetilde{N}_{\mathrm{AW}}(r, a_\nu) + S_{\log}(r, \varepsilon; f) \tag{8}$$

where $S_{\log}(r, \varepsilon; f) = O\big((\log r)^{\sigma_{\log}-1+\varepsilon}\big)$, $\widetilde{N}_{\mathrm{AW}}(r, f)$ *and* $\widetilde{N}_{\mathrm{AW}}(r, a_\nu)$ *are the* $\mathrm{AW}-$ *counting functions.*

We deduce

$$\sum_{a\in\widehat{\mathbb{C}}} (\delta(a) + \theta_{\mathrm{AW}}(a)) \leq \sum_{a\in\widehat{\mathbb{C}}} \Theta_{\mathrm{AW}}(a) \leq 2,$$

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Rational AW-Picard Deficienies

- $$f_{\frac{1}{2}}(x) = (e^{i\theta}, e^{-i\theta}; q)_\infty (q^2 e^{i\theta}, q^2 e^{-i\theta}; q^3)_\infty, \quad \Theta_{\mathrm{AW}}(0) = 1/2.$$

- $$f_{\frac{2}{3}}(x) = (e^{i\theta}, e^{-i\theta}; q^4)_\infty (q e^{i\theta}, q e^{-i\theta}; q^4)_\infty \cdot (q^2 e^{i\theta}, q^2 e^{-i\theta}; q^4)_\infty, \qquad \Theta_{\mathrm{AW}}(0) = 2/3$$

- $$f_{\frac{1}{n}}(x) = \prod_{k=0}^{n-1} (q^{2k} e^{i\theta}, q^{2k} e^{-i\theta}; q^{2n-1})_\infty, \quad \Theta_{\mathrm{AW}}(0) = 1/n.$$

- $$f_{\frac{n-1}{n}}(x) = \prod_{k=0}^{n-1} (q^k e^{i\theta}, q^k e^{-i\theta}; q^{n+1})_\infty, \qquad \Theta_{\mathrm{AW}}(0) = (n-1)/n,$$

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The Askey-Wilson "Constants"

- This terminology is due to Mourad Ismal.
- Let f lies in the kernel of the AW-operator. Then there exists a non-negative integer k and complex numbers $a_1, \cdots, a_k$ and $b_1, \cdots, b_k$, $C \neq 0$ such that

$$f(x) = C \prod_{j=1}^{k} \frac{(a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty (q/a_j e^{i\theta}, q/a_j e^{-i\theta}; q)_\infty}{(b_j e^{i\theta}, b_j e^{-i\theta}; q)_\infty (q/b_j e^{i\theta}, q/b_j e^{-i\theta}; q)_\infty}$$

Kernel identities

Theorem (C.& Feng (2018))

Given positive integer k and complex numbers $a_j,\ C_j,\ j = 1, 2, \cdots k$, there exist complex numbers b and C such that

$$\sum_{j=1}^{k} C_j\,(a_j e^{iz}, a_j e^{-iz};\, q)_\infty (q/a_j e^{iz},\, q/a_j e^{-iz};\, q)_\infty$$

$$= C\,(be^{iz},\, be^{-iz};\, q)_\infty (q/be^{iz},\, q/be^{-iz};\, q)_\infty .$$

Theta functions identities

- $$\vartheta_4(z/2) = (q^2,\, q^2)_\infty\, (q\, e^{iz},\, q\, e^{-iz};\, q^2)_\infty$$

- $$\vartheta_3(z/2) = (q^2;\, q^2)_\infty\, (-q\, e^{iz},\, -q\, e^{-iz};\, q^2)_\infty.$$

- $$C_1\, \vartheta_4^2(z) + C_2\, \vartheta_2^2(z) = C\, \vartheta_3^2(z)$$

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An Example

- Consider the Jacobian (elliptic) theta functions:

$$f(x) = \Theta_4(2\cos\theta,\ q) = 2\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\cos(2n\theta)$$
$$= \left(q^2,\ qe^{2i\theta},\ qe^{-2i\theta};\ q^2\right)_\infty,$$

and

$$g(x) = \Theta_3(\cos 2\theta,\ q) = 2\sum_{-\infty}^{\infty} q^{n^2}\cos(2n\theta)$$
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- Then the function

$$F(x) = \frac{f(x)}{g(x)}$$

has $\{0,\ \infty\}$ to be **Askey-Wilson-Picard exceptional values**, and there are no other zeros and poles of F.

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Summary

- We have reviewed on recent development on function theory related to difference operators
- Askey-Wilson type Nevanlinna theory
- Interpreted the infinite Zeros/poles sequences that lie on particular orbit have $\Theta_{\mathrm{AW}}(\cdot) = 1$ so they are like missing in the Nevanlinna sense,
- Future directions may include:
  1. Value distribution results vs special function identities
  2. Applications to difference equations
  3. Missing piece: Laurent series w. r. t. different bases?
  4. Relations with interpolation theory
  5. Any modular proof?

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